Low-Energy Theorems for e^4 Compton Scattering Amplitudes

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Unitarity and fixed-momentum-transfer dispersion relations are used to evaluate the contribution to the e^4 Compton amplitude from the intermediate state of target plus soft photon. It is found that the leading term goes like $\omega^2 \ln\omega$ for scattering of photons of low frequency (ω) by either spin-0 or spin-$\frac{1}{2}$ targets. The e^4 amplitudes up to this order ($\omega^2 \ln\omega$) are shown to be structure-independent as a consequence of the usual Compton-scattering low-energy theorems, which have been proven rigorously only in order e^2.

I. INTRODUCTION

IT has been known[1,2] for a long time that when the scattering amplitudes of low-energy photons by spin-0 and spin-$\frac{1}{2}$ targets are expanded in a power series of the photon frequency (ω), the entire zeroth- and first-order coefficients can be expressed in terms of the static properties of the target system: the electric charge and the magnetic moment of the particle. Recently these theorems have been further extended, to scattering involving "charged" photons,[3] also to parts of the $O(\omega^2)$ amplitude[4] and to cases of higher-spin targets.[5] The derivations of all these theorems rest on the crucial assumption that the single-particle intermediate state[6] is separated from the multiparticle states by a finite energy gap. Thus the presence of intermediate states made up of a target particle and soft photons will invalidate this assumption. Consequently, while the usual proofs of low-energy theorems are valid to all orders of strong interactions, their validity has been demonstrated only to the lowest order in electromagnetism.

An equivalent way of stating the same problem is to say that the original power-series expansion in photon frequencies may not be valid to the desired order of ω in the presence of radiative corrections. Intermediate states of target particle plus soft photons bring about the well-known infrared-divergent terms, as well as finite terms proportional to $\omega^n \ln\omega$, in the elastic Compton scattering amplitude. Clearly the presence of any terms proportional to $\omega \ln\omega$ will invalidate the usual low-energy theorems for the linear-frequency part of the amplitude.

The physical basis of the infrared divergence has long been clear and the divergence, as is to be expected, cancels if one adds the corresponding cross section for emission of soft photons in the scattering process.[7] These divergent terms can be calculated in a structure-independent way, at least in the first few orders of α, by the usual perturbation method.[8] They are always multiplied by an angular factor which vanished in the forward direction, corresponding to the physical fact that there will be no radiation by the charged target particle when it is not accelerated. Thus we expect that there will be no modification by *infrared-divergent* terms of the low-energy theorems in the forward direction. As for nonforward amplitudes, modifications by these terms will clearly depend on the energy resolution of the experimental setup.[9] Since these features are well known, we will concentrate on the *finite* terms that are proportional to $\omega^n \ln\omega$.

As a first step, we will investigate the leading contribution to Compton scattering amplitude from the intermediate state of target plus *one* soft photon. Our method will be first to compute the contribution to the absorptive part of the amplitude up to, and including, $O(\omega^2)$. This can be done since the absorptive amplitude involves only an angular integration over a product of two e^2 Compton scattering amplitudes of low-energy photons, each of which can be evaluated up to $O(\omega)$ by the usual low-energy theorems. Phase-space factors are of $O(\omega)$. Fixed-t dispersion relations are then used to obtain the dispersive parts. In this way we obtain for the e^4 Compton scattering amplitude with either spin-0

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[1] W. Thirring, Phil. Mag. **41**, 1193 (1950).

[2] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* **96**, 1433 (1954).

[3] M. A. B. Bég, Phys. Rev. Letters **17**, 333 (1966).

[4] V. Singh, Phys. Rev. Letters **19**, 730 (1967).

[5] A. Pais, Phys. Rev. Letters **19**, 544 (1967); Nuovo Cimento **53**, 433 (1968); see also, K. Bardakci and H. Pagels, Phys. Rev. **166**, 1783 (1968).

[6] It is possible that in certain instances there are more than one single-particle intermediate states [for example, the Λ, Σ^0 states in the $SU(3)$ limit]. This fact will not affect the validity of the usual derivations. Here we have assumed, for simplicity, that there is only one such single-particle state.

[7] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

[8] For a detailed discussion of the infrared-divergence problem in the context of modern perturbation theory see J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955), Chap. 16.

[9] Possible modifications of the Thomson low-energy theorem by the infrared-divergent terms have been discussed by W. Thirring [*Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958), p. 190], and by J. D. Bjorken and S. D. Drell [*Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1964), p. 363]. With the corresponding soft-photon emission cross section being included, the Compton scattering differential cross section has its zero-frequency limit as

$$\frac{\alpha^2}{m^2}(\mathbf{e}_0\cdot\mathbf{e}^2)\left[1+\frac{2\alpha}{3\pi}\left(\frac{\omega}{m}\right)^2(1-\cos\theta)\ln|\Delta\omega/m|\right]\Big|_{\omega\to 0},$$

where $\Delta\omega$ is the energy resolution of the detector. Qualitatively we expect that with the finite resolution the limit $\omega\to 0$ can at best, in a real experiment, be taken to mean $\omega\to\Delta\omega$. This will lead to a correction factor $\simeq(2\alpha/3\pi)(\Delta\omega/m)^2(1-\cos\theta)\ln|\Delta\omega/m|$, which should be a very small number compared to the leading constant term.

or spin-$\frac{1}{2}$ targets:

$$\frac{\alpha}{3\pi}\left(\frac{\alpha}{m}\right)\left[\left(\frac{\omega}{m}\right)^2(3+\cos\theta)(\mathbf{e}_0\cdot\mathbf{e})-\frac{2}{m^2}(\mathbf{e}_0\cdot\mathbf{k})(\mathbf{e}\cdot\mathbf{k}_0)\right]$$
$$\times\ln\omega+\text{IRD}+\text{terms of }O(\omega^2),\quad(1.1)$$

where $\mathbf{e}_0$ ($\mathbf{e}$) and $\mathbf{k}_0$ ($\mathbf{k}$) are, respectively, the polarization and momentum vectors of the initial (final) photon, and $\cos\theta$ is the scattering angle in the rest frame of the target. IRD stands for infrared-divergent terms. We are unable to compute unambiguously their explicit form by our dispersion method. According to perturbation calculations,

$$\text{IRD}=\frac{2\alpha}{3\pi}\left(\frac{\alpha}{m}\right)\left(\frac{\omega}{m}\right)^2(1-\cos\theta)\ln\omega_{\min}(\mathbf{e}_0\cdot\mathbf{e})$$
$$+\text{higher-order terms.}$$

The cross product of the e^4 amplitude in Eq. (1.1) and the Thomson amplitude, $-(\alpha/m)(\mathbf{e}_0\cdot\mathbf{e})$, gives rise to a leading contribution to the e^6 differential cross section,

$$d\sigma_6=d\Omega\frac{2\alpha}{3\pi}\left(\frac{\alpha}{m}\right)^2\left(\frac{\omega}{m}\right)^2[(\cos^3\theta-3\cos^2\theta-3\cos\theta-3)\ln\omega$$
$$+(1-\cos\theta)(1+\cos^2\theta)\ln\omega_{\min}]+O(\omega^2).\quad(1.2)$$

This expression agrees with the low-energy limits of the exact relativistic results calculated in perturbation theory by Corinaldesi and Jost for a spin-0 target,[10] and by Brown and Feynman for an electron target.[11] *With our approach the results (1.1) and (1.2) are proven to be valid to all orders of strong interactions.*

We observe that the e^4 amplitude for a spin-$\frac{1}{2}$ target in Eq. (1.1) does not contain, up to $O(\omega^2)$, any spin-dependent terms. Indeed it will be shown in Sec. III that all spin-dependent terms are at least of the order of $\omega^3\ln\omega$. Thus the usual low-energy theorems are correct in order e^4 except in the nonforward directions where there will be an expected modification by the infrared-divergent terms.[9] We note parenthetically that Brown and Feynman[11] are only concerned with spin-average cross sections. They summed over electron spins at the very beginning of their calculation.

Knowing the result for the e^4 amplitude and the fact that there is no infrared divergence in the forward amplitudes, we go on to consider the possibility that the usual low-energy theorems for forward Compton amplitudes are exact to all orders of the electromagnetic coupling. There are only two independent forward amplitudes (f_i) for a spin-$\frac{1}{2}$ target,

$$f_1(\omega)(\mathbf{e}_0\cdot\mathbf{e})+if_2(\omega)\boldsymbol{\sigma}\cdot(\mathbf{e}_0\times\mathbf{e}).\quad(1.3)$$

Their absorptive parts (a_i) are related, through the optical theorem, to the total cross sections:

$$a_1(\omega)=(\omega/4\pi)\tfrac{1}{2}[\sigma_P(\omega)+\sigma_A(\omega)],\quad(1.4)$$

$$a_2(\omega)=(\omega/4\pi)\tfrac{1}{2}[\sigma_P(\omega)-\sigma_A(\omega)],\quad(1.5)$$

where σ_P is the total cross section for circularly polarized photons with their helicity parallel to the target spin; σ_A, antiparallel. The usual low-energy theorem states that in the low-frequency limit the leading terms of these amplitudes are

$$f_1=-e^2/m,\quad(1.6)$$

$$f_2=2\omega\kappa^2(e/2m)^2,\quad(1.7)$$

where κ is the anomalous magnetic moment of the target particle measured in units of $(e/2m)$. The theorem for f_1 in Eq. (1.6) should be correct to all orders of α. On the other hand, terms that might be possible, terms proportional to $\omega\ln\omega$, coming from the intermediate state of the target plus soft photons, would spoil the theorem for f_2 as expressed in Eq. (1.7). These terms are shown explicitly to be absent to the e^4 order. We now argue that they should be absent to all orders of electromagnetic coupling because the existence of such terms will be shown to imply physically unreasonable features for the total cross sections. We note that such an $\omega\ln\omega$ term in f_2 would imply a term proportional to ω in the forward absorptive amplitude a_2. It would then imply, through Eq. (1.5), that σ_P and σ_A would have to be different in the *zero*-frequency limit. This would be a totally unphysical feature since cross sections for scattering zero-frequency photons cannot possibly be dependent on the spin orientation of the target. Of course there remains the possibility that there would be other types of terms, besides these logarithms,[12] which could invalidate the usual low-energy theorems. It is our conjecture that all the special functions involved will reduce to simple functions and logarithms when their arguments take on the nonrelativistic limit. The Spence function is a well-known example.[11] Thus it seems plausible, although we have offered a proof only in order e^4, that the low-energy theorem for the forward amplitude Eq. (1.7) is exact to all orders in electromagnetism.

In Sec. II, we will review briefly the usual proofs of low-energy theorems. These procedures are rigorously valid only in the lowest order of α. Kinematical preliminaries will also be presented in this section. In Sec. III, the new low-energy theorems for e^4 Compton amplitudes will be derived, first for spin 0, then for spin $\frac{1}{2}$.

While this work was being completed, I learned that Dr. Roy and Dr. Singh had obtained similar low-energy theorems for e^4 Compton amplitudes via fixed-angle dispersion relations.[13,14]

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[10] F. Corinaldesi and R. Jost, Helv. Phys. Acta **21**, 183 (1948).

[11] L. M. Brown and R. P. Feynman, Phys. Rev. **85**, 231 (1952).

[12] Terms proportional to $\omega[\ln\omega]^N$ with $N>1$ can be ruled out in a similar manner. They would imply the total cross sections to be divergent in the zero-frequency limit.

[13] S. M. Roy and V. Singh, Phys. Rev. Letters **21**, 681 (1968). I am grateful to Dr. M. A. B. Bég, Dr. A. Pais, and Dr. J. Pasupathy for calling my attention to this work.

[14] *Note added in proof.* It has been brought to my attention that L. D. Soloviev [Nucl. Phys. **64**, 657 (1965)] has investigated the problem of the low-energy expansion of amplitudes involving photons to all orders in the electromagnetism.

II. e^2 LOW-ENERGY THEOREMS

We here review briefly the usual proofs of low-energy theorems. Although there exist a number of different approaches, all of them have the same basic inputs: Lorentz invariance, gauge invariance, and the "regularity assumption." The content of the last input will become clear as we proceed. We shall adopt an approach that is in substance a manifestly covariant formulation of the method originally used by Low.[15]

Consider the scattering of photons with initial (final) four-momentum and polarization k_0, $e_{0\mu}$ (k,e_λ), respectively, by a target of arbitrary spin with initial (final) momentum $p_0(p)$; the tensorial amplitude $T_{\mu\lambda}$ is related to the S matrix by

$$\langle p,k|S-1|p_0,k_0\rangle=-i(2\pi)^4\delta(p_0+k_0-p-k)\times NN_0\frac{1}{(2\pi)^3}\left(\frac{1}{4\omega\omega_0}\right)^{1/2}e_{0\mu}e_\lambda T_{\mu\lambda}(p,k;p_0,k_0)\,, \quad (2.1)$$

where N_0 and N are the appropriate normalization factors for the initial and final states of the target particle. $T_{\mu\lambda}$ is related to the matrix element of the electromagnetic current J by

$$T_{\mu\lambda}(p,k;p_0,k_0)=-i\frac{1}{NN_0}\int dx\,e^{-i(k+k_0)\cdot x/2}\times\theta(x)\langle p|[J_\lambda(\tfrac{1}{2}x),J_\mu(-\tfrac{1}{2}x)]|p_0\rangle+\text{ETC}\,, \quad (2.2)$$

where ETC stands for possible equal-time commutator factors, including the Schwinger terms. Crossing symmetry implies that

$$T_{\mu\lambda}(p,k;p_0k_0)=\epsilon T_{\lambda\mu}(p,-k_0;p_0,-k)\,; \quad (2.3)$$

$\epsilon=+1$ for the dispersive part, $\epsilon=-1$ for the absorptive part. Gauge invariance requires that

$$k_{0\mu}T_{\mu\lambda}=k_\lambda T_{\mu\lambda}=0\,. \quad (2.4)$$

We separate out from the full amplitude a contribution corresponding to one made by the single-particle state $U_{\mu\lambda}$:

$$T_{\mu\lambda}=U_{\mu\lambda}+E_{\mu\lambda}\,. \quad (2.5)$$

Write

$$\begin{aligned}T_{\mu\lambda}&=U_{\mu\lambda}+c_{\mu\lambda}+E_{\mu\lambda}-c_{\mu\lambda}\\&\equiv U_{\mu\lambda}'+E_{\mu\lambda}'\,,\end{aligned} \quad (2.6)$$

$c_{\mu\lambda}$ being nonsingular and such that $U_{\mu\lambda}'$ is gauge-invariant, $k_{0\mu}U_{\mu\lambda}'=k_\lambda U_{\mu\lambda}'=0$. It then follows that the remaining part also satisfies the conditions

$$k_{0\mu}E_{\mu\lambda}'=0\,, \quad (2.7a)$$

$$k_\lambda E_{\mu\lambda}'=0\,. \quad (2.7b)$$

Differentiating the above two equations with respect to k_0 and k, we get

$$E_{\mu\lambda}'=-k_{0\nu}\frac{\partial}{\partial k_{0\mu}}E_{\nu\lambda}'\,, \quad (2.8a)$$

$$E_{\mu\lambda}'=-k_\nu\frac{\partial}{\partial k_\lambda}E_{\mu\nu}'\,. \quad (2.8b)$$

Treating k_0 and k as independent variables, we have, from Eqs. (2.8a) and (2.8b),

$$E_{\mu\lambda}'=k_{0\nu}k_\rho\frac{\partial}{\partial k_{0\mu}}\frac{\partial}{\partial k_\lambda}E_{\nu\rho}'\,. \quad (2.8c)$$

Thus, the existence of the limit

$$\left.\frac{\partial}{\partial k_{0\mu}}\frac{\partial}{\partial k_\lambda}E_{\nu\rho}'\right|_{k_0\to0,k\to0}$$

(the "regularity assumption")[16] implies that $E_{\mu\lambda}'$ is of the order ω_0 *and* of the order ω. For physical scattering, ω_0 and ω being related through conservation of momentum, $E_{\mu\lambda}'$ is then of the order ω^2. The low-energy theorem is established:

$$T_{\mu\lambda}=U_{\mu\lambda}'+O(\omega^2)\,. \quad (2.9a)$$

To the lowest order in e the "regularity assumption" is clearly correct. Indeed, to this order, all the derivatives of $E_{\mu\lambda}'$ with respect to k and k_0 exist at the zero-frequency limit. On the other hand, since contributions from intermediate states of target plus photons give rise to cuts extending to the point ($\omega_0=\omega=0$) where the amplitude is being expanded, the "regularity assumption" [hence the low-energy theorem as stated in Eq. (2.9a)] is open to question in the presence of higher-order radiative corrections. We shall see, in our explicit calculation of the e^4 Compton amplitude, that the "regularity assumption" as stated here is indeed not valid to this order (i.e., the double derivative in question is singular in this limit). But, as one of our main results, the leading contribution due to the intermediate target plus one soft-photon state will be shown to be proportional to $\omega^2\ln\omega$. Consequently the usual low-energy theorem, Eq. (2.9a), will only be modified to read, taking into account the expected infrared divergence, as

$$T_{\mu\lambda}=U_{\mu\lambda}'+O(e^4\omega^2\ln\omega)+\text{IRD}\,. \quad (2.9b)$$

Thus to obtain the low-energy behavior of e^2 Compton scattering amplitude, our principal task will be, according to Eq. (2.9a), to compute the gauge-invariant contribution from the one-particle state. We will work out the familiar results for the case of spin-0 and spin-$\frac{1}{2}$ targets. Our purpose is simply to set up machinery for the next section, where in writing out the e^4 low-energy theorems we need to evaluate the gauge-

[15] We follow closely to the work of R. Jackiw, Phys. Rev. **168**, 1623 (1968); see, also, Bardakci and Pagels (Ref. 5).

[16] See, for example, J. S. Bell, Nuovo Cimento **52**, 635 (1967).

invariant contribution coming from the state of target plus one soft photon. For convenience we will often refer to the spin-0 target as a pion and to the spin-$\frac{1}{2}$ target as a nucleon.

A. Spin-0 Case

The contribution of the single-pion intermediate state to the absorptive part of the amplitude can be written down immediately by Eq. (2.2) (here s, t, u denote the usual Mandelstam variables):

$$-\pi e^2[\delta(s-m^2)\theta(s-m^2)(2p_0+k_0)_\mu(2p+k)_\lambda - \delta(u-m^2)\theta(u-m^2)(2p-k_0)_\mu(2p_0-k)_\lambda], \quad (2.10)$$

where we have used, for the electromagnetic matrix element,

$$\langle p|J_\lambda(0)|p_0\rangle = \frac{i}{(2\pi)^3}\left(\frac{1}{4\epsilon_0\epsilon}\right)^{1/2} e(p+p_0)_\lambda \quad \text{for } (p-p_0)^2=0,$$

ϵ_0 and ϵ being the energy variables of the target. To obtain the corresponding amplitude a fixed-t dispersion relation may be formally applied:

$$f(s,t,u)=\rho(t)+\frac{1}{\pi}\int_{s_0}\frac{\bar{f}_s(s',t)ds'}{s'-s}+\frac{1}{\pi}\int_{u_0}\frac{\bar{f}_u(u',t)du'}{u'-u}, \quad (2.11)$$

where $\bar{f}_s$ and $\bar{f}_u$ are the appropriate s- and u-channel absorptive amplitudes. In our case they can be read off from Eq. (2.10). $\rho(t)$ stands for possible subtraction terms and incidentally the ETC of Eq. (2.2) will contribute to such a term in the amplitude. The requirement of gauge invariance then fixes $\rho(t)$ to be $-2e^2\delta_{\lambda\mu}$. We obtain the low-energy theorem for $\gamma\pi$ scattering:

$$T_{\lambda\mu}=e^2\frac{(2p_0+k_0)_\mu(2p+k)_\lambda}{s-m^2}+e^2\frac{(2p-k_0)_\mu(2p_0-k)_\lambda}{u-m^2}-2e^2\delta_{\lambda\mu}+O(\omega^2). \quad (2.12)$$

B. Spin-$\frac{1}{2}$ Case

The electromagnetic current operator between nucleon states is

$$\langle p|J_\lambda(0)|p_0\rangle=\frac{i}{(2\pi)^3}\left(\frac{m^2}{\epsilon_0\epsilon}\right)^{1/2} \times e\bar{u}(p)\left[\gamma_\lambda-\left(\frac{\kappa}{2m}\right)\sigma_{\lambda\nu}(p-p_0)_\nu\right]u(p_0)$$

for $(p-p_0)^2=0$. In the same way as seen in the case of a spin-0 target, we get a low-energy theorem for γN scattering, in covariant form:

$$T_{\mu\lambda}=-e^2\bar{u}(p)\left\{\left[\gamma_\lambda+\left(\frac{\kappa}{2m}\right)\sigma_{\lambda\nu}k_\nu\right]\frac{i(p_0+k_0)\cdot\gamma-m}{s-m^2}\left[\gamma_\mu-\left(\frac{\kappa}{2m}\right)\sigma_{\mu\nu}k_{0\nu}\right] +\left[\gamma_\mu-\left(\frac{\kappa}{2m}\right)\sigma_{\mu\nu}k_{0\nu}\right]\frac{i(p_0-k)\cdot\gamma-m}{u-m^2}\left[\gamma_\lambda+\left(\frac{\kappa}{2m}\right)\sigma_{\lambda\nu}k_\nu\right]\right\}u(p_0)+O(\omega^2). \quad (2.13)$$

In the rest frame of the target nucleon with transverse photons, the above expression reduced to a matrix taken between Pauli spinors χ,

$$e_{0i}e_jT_{ij}=\chi^\dagger\left[-\frac{e^2}{m}(\mathbf{e}_0\cdot\mathbf{e})-2i\omega\left(\frac{e}{2m}\right)^2(1+2\kappa)\boldsymbol{\sigma}\cdot(\mathbf{e}_0\times\mathbf{e})+2i\omega\left(\frac{e}{2m}\right)^2(1+\kappa)^2\boldsymbol{\sigma}\cdot(\mathbf{e}_0\times\hat{k}_0)\times(\mathbf{e}\times\hat{k}) +2i\omega\left(\frac{e}{2m}\right)^2(1+\kappa)[\boldsymbol{\sigma}\cdot(\mathbf{e}_0\times\hat{k}_0)(\hat{k}_0\cdot\mathbf{e})-\boldsymbol{\sigma}\cdot(\mathbf{e}\times\hat{k})(\hat{k}\cdot\mathbf{e}_0)]\right]\chi+O(\omega^2) \quad (2.14)$$

with $i(j)=1, 2, 3$.

III. e^4 LOW-ENERGY THEOREMS

As we discussed in Sec. II, the intermediate state of target particle plus soft photons (being degenerate with the single-particle state in the zero-photon-energy limit) may contribute a piece to the amplitude of orders *less* than ω^2 in the low-frequency limit. In this section we will study the simplest case: contributions from the state of target plus *one* soft photon. Our approach here is similar to that used in Sec. II, where the single-particle-state contribution is calculated.

A. Spin-0 Case

The dispersive part of the tensorial amplitude $T_{\mu\lambda}$ which satisfies the requirements of crossing symmetry, time-reversal invariance, and Hermiticity of the currents can be expressed in terms of the invariant basis as

$$T_{\mu\lambda}=F_1\delta_{\mu\lambda}+F_2k_\mu k_{0\lambda}+F_3P_\mu P_\lambda+F_4(k_\mu P_\lambda+P_\mu k_{0\lambda}), \quad (3.1)$$

$P=p+p_0$, and $F_i=F_i(s,t,u)$. We have dropped all terms that are proportional to either $k_{0\mu}$ or k_λ. Gauge invariance requires that

$$(u-s)F_4+tF_2+2F_1=0\,, \tag{3.2a}$$

$$(u-s)F_3+tF_4=0\,. \tag{3.2b}$$

Just as in the case of the single-particle state, we will first compute contributions to the absorptive amplitude from the pion plus one soft photon state $|p',k'\rangle$. By Eq. (2.2) we have

$$\begin{aligned}M_{\mu\lambda}=&-\frac{1}{8\pi^2}\int\frac{d^3p'}{2\epsilon'}\frac{d^3k'}{2\omega'}[\delta(p+k-p'-k')T_{\lambda\nu}^*(p,k;p',k')T_{\mu\nu}(p_0,k_0;p',k')\\&-\delta(p-k_0-p'-k')T_{\mu\nu}^*(p,-k_0;p',k')T_{\lambda\nu}(p_0,-k;p',k')]\\=&-\frac{1}{32\pi^2}\bigg[\left(\frac{s-m^2}{2s}\right)\theta(s-m^2)\int d\Omega' T_{\lambda\nu}^*(p,k;p',k')T_{\mu\nu}(p_0,k_0;p',k')\\&-\left(\frac{u-m^2}{2u}\right)\theta(u-m^2)\int d\Omega' T_{\mu\nu}^*(p,-k_0;p',k')T_{\lambda\nu}(p_0,-k;p',k')\bigg]\,,\end{aligned} \tag{3.3}$$

where $d\Omega'$ is an element of the solid angle contained by $\hat{k}'$ in the barycentric frame of reference. The T's are e^2 Compton scattering amplitudes with all photons on mass shell. Since we are only interested in the region where $k_0\to 0$ and $k\to 0$, we are entitled to apply the low-energy theorem as expressed in Eq. (2.12) to these T's. Clearly, there is no question that coefficients of the θ functions in $M_{\mu\lambda}$ can be expanded in a power series of ω, since power-series expansion is known to be valid for each of the T's. We can compute $(T^*T)_{\mu\lambda}$ up to $O(\omega)$ and hence $M_{\mu\lambda}$ up to $O(\omega^2)$, the phase-space factors $(s-m^2)/s$ and $(u-m^2)/u$ being of order ω. When expressed in terms of invariant bases of Eq. (3.1), the absorptive amplitudes M_i corresponding to F_i are computed to be

$$\begin{aligned}M_1=&-\frac{e^4}{4\pi}\left\{\theta(s-m^2)\left[\frac{2}{3}\left(\frac{s-m^2}{m^2}\right)-\frac{2}{3}\left(\frac{s-m^2}{m^2}\right)^2+O(\omega^3)\right]-\theta(u-m^2)\left[\frac{2}{3}\left(\frac{u-m^2}{m^2}\right)-\frac{2}{3}\left(\frac{u-m^2}{m^2}\right)^2+O(\omega^3)\right]\right\}\,,\\M_2=&-\frac{e^4}{4\pi}\left[\theta(s-m^2)\left(\frac{4}{3m^2}+O(\omega)\right)-\theta(u-m^2)\left(\frac{4}{3m^2}+O(\omega)\right)\right]\,,\\M_3=&-\frac{e^4}{4\pi}\left\{\theta(s-m^2)\left[\frac{1}{3m^2}\left(\frac{t}{s-m^2}\right)-\frac{t}{3m^4}+O(\omega^3)\right]-\theta(u-m^2)\left[\frac{1}{3m^2}\left(\frac{t}{u-m^2}\right)-\frac{t}{3m^4}+O(\omega^3)\right]\right\}\,,\\M_4=&-\frac{e^4}{4\pi}\left\{\theta(s-m^2)\left[\frac{2}{3m^2}-\frac{2}{3m^2}\left(\frac{s-m^2}{m^2}\right)+\frac{1}{3m^2}\left(\frac{t}{s-m^2}\right)+O(\omega^2)\right]\right.\\&\left.+\theta(u-m^2)\left[\frac{2}{3m^2}-\frac{2}{3m^2}\left(\frac{u-m^2}{m^2}\right)+\frac{1}{3m^2}\left(\frac{t}{u-m^2}\right)+O(\omega^2)\right]\right\}\,.\end{aligned} \tag{3.4}$$

Assuming that each $F_i(s,t,u)$ satisfies the simple dispersion relation of Eq. (2.11), we then have from the threshold contribution,[17]

$$\begin{aligned}F_1=&-\frac{e^4}{4\pi^2}\left[-\frac{2}{3}\frac{t}{m^2}-\frac{1}{3}\left(\frac{u-s}{m^2}\right)^2\right]\ln|s-m^2|+\rho_1(t)+O(\omega^2)\,,\\F_2=&-\frac{e^4}{4\pi^2}\left(\frac{8}{3m^2}\right)\ln|s-m^2|+\rho_2(t)+O(\omega^0)\,,\\F_3=&-\frac{e^4}{4\pi^2}\left[\frac{4}{3m^2}\left(\frac{t}{s-u}\right)^2-\frac{2}{3}\frac{t}{m^4}\right]\ln|s-m^2|+\frac{e^4}{3\pi^2m^2}\left(\frac{t}{s-u}\right)^2\ln\lambda^2+\rho_3(t)+O(\omega^2)\,,\\F_4=&-\frac{e^4}{4\pi^2}\left[\frac{2}{3m^2}\left(\frac{u-s}{m^2}\right)-\frac{4}{3m^2}\left(\frac{t}{u-s}\right)\right]\ln|s-m^2|-\frac{e^4}{3\pi^2m^2}\left(\frac{t}{s-u}\right)^2\ln\lambda^2+\rho_4(t)+O(\omega)\,,\end{aligned} \tag{3.5}$$

[17] We note that $\ln|s-m^2|=\ln|u-m^2|+O(\omega)=O(\ln\omega)$. Any confusion as to the proper crossing properties of the amplitudes in Eqs. (3.4) and (3.5) should be cleared by the simple observation that an dispersive amplitude $\sim\ln\omega^2$ (even under crossing) corresponds to an absorptive amplitude $\sim\pi\epsilon(\omega)$ (odd under crossing). $\epsilon(\omega)=+1$ for $\omega>0$; $\epsilon(\omega)=-1$ for $\omega<0$. The ϵ function comes about because the physical amplitude is defined as the limit to be approached from "above" ($\omega+i\epsilon$, $\epsilon\to 0$) of the cuts in the complex ω plane. Consequently the contours have to be taken in opposite directions for the evaluation of discontinuities of the right- and left-hand cuts.

λ being the fictitious photon mass. The infrared-divergent terms come about in F_3 and F_4 because we used fixed-t dispersion relations and there were $t/(s-m^2)$ and $t/(u-m^2)$ terms in M_3 and M_4. As it should be, terms proportional to $\ln|s-m^2|$ automatically satisfy the gauge-invariance requirements of Eq. (3.2). For the rest of the amplitudes Eq. (3.2) implies that

$$\rho_3(t)=\rho_4(t)=0$$

and

$$(e^4/3\pi^2m^2)t\ln\lambda^2-t\rho_2(t)-2\rho_1(t)=0.$$

The usual perturbation result as stated in the Introduction corresponds to the solution

$$\rho_2(t)=0$$

and

$$\rho_1(t)=(e^4/6\pi^2m^2)t\ln\lambda^2.$$

In the rest frame of the target pion, the contribution from terms of F_3 and F_4 to the amplitude T_{ij} will be of the order of higher than ω^2; hence only F_1 (δ_{ij}) and F_2 (k_ik_{0j}) amplitudes appear in Eq. (1.1). They are of the order $\omega^2\ln\omega^2$. Actually in this case of a spin-0 target we could have anticipated this general feature. Since the absorptive amplitude in Eq. (3.3) contains terms proportional to $\omega\epsilon(\omega)$ and $\omega^2\epsilon(\omega)$ [$\epsilon(\omega)=+1$ for $\omega>0$; -1 for $\omega<0$], the corresponding dispersive parts should be proportional to $\omega\ln\omega^2$ and $\omega^2\ln\omega^2$. The $\omega\ln\omega^2$ terms cancel as required by crossing symmetry.

B. Spin-$\frac{1}{2}$ Case

We will now treat the more interesting, but algebraically much more involved, case of Compton scattering with spin-$\frac{1}{2}$ targets. Unlike the case of spin 0, there is no simple argument within the context of our calculation that we can use to rule out the possible $\omega\ln\omega$ terms since there are spin-dependent terms that are odd under crossing. If such terms exist, the usual derivation of low-energy theorems for the linear-frequency part of the amplitude would be invalidated.

The approach we are going to take will be the same as that used in $\gamma\pi$ scattering. But with the experience and understanding gained from the spin-0 case, we will be able to simplify the algebraic problem considerably. We have learned that the amplitudes coming from dispersion integrals over target plus soft photon cuts will be proportional to $\ln\omega$ and hence must satisfy the gauge-invariance conditions of Eqs. (3.2) among themselves. Consequently in our evaluation of the absorptive parts,

$$M_{\mu\lambda}=-\frac{1}{8\pi^2}\int\frac{m}{\epsilon'}d^3p'\frac{d^3k'}{2\omega'}[\delta(p+k-p'-k')\times T_{\lambda\nu}^*(p,k;p',k')T_{\mu\nu}(p_0,k_0;p',k')-\delta(p-k_0-p'-k')T_{\mu\nu}^*(p,-k_0;p',k')\times T_{\lambda\nu}(p_0,-k;p',k')],\quad(3.6)$$

although we must still use the Lorentz- and gauge-invariant amplitudes of Eq. (2.13) for each of the T's, the product $(T^*T)_{\lambda\mu}$ may be computed in any gauge and frame of reference we choose. Thus in the rest frame of the target particle ($\mathbf{p}_0=0$) with transverse photons we have, after a rather lengthy calculation,

$$\begin{aligned}M_{ij}=&-\frac{e^4}{8\pi m}\theta(s-m^2)\left\{\left[\frac{2}{3}\left(\frac{s-m^2}{m^2}\right)-\frac{2}{3}\left(\frac{s-m^2}{m^2}\right)^2\right]\delta_{ij}+\frac{4}{3m^2}k_ik_{0j}+O(\omega^3)\right\}\\&+\frac{e^4}{8\pi m}\theta(u-m^2)\left\{\left[\frac{2}{3}\left(\frac{u-m^2}{m^2}\right)-\frac{2}{3}\left(\frac{u-m^2}{m^2}\right)^2\right]\delta_{ij}+\frac{4}{3m^2}k_ik_{0j}+O(\omega^3)\right\}\\&+\text{spin-dependent terms of }O(\omega^2)\end{aligned}\quad(3.7)$$

with $i(j)=1, 2, 3$. Through fixed-t dispersion relations we then get the result for the $(e^4)\gamma N$ Compton amplitude as stated in Eq. (1.1). We did not explicitly calculate the spin-dependent terms of $O(\omega^2)$ in the absorptive amplitude. Their corresponding dispersive amplitudes, being proportional to $\omega^2\ln\omega^2$, should cancel as required by crossing symmetry, because all spin-dependent amplitudes are odd under crossing. We note that in each of the two terms in Eq. (3.7) there are no $O(\omega)$ spin-dependent terms; they all cancel in the explicit computation. Consequently the result is that the e^4 spin-dependent amplitudes of nucleon Compton scattering are of the order of $\omega^3\ln\omega^2$.

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